Risk and Ambiguity

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Introduction

Risk measures theory:

- Provide a *mathematical definition* of measures of risks.
- Present and justify a *unified* framework for the analysis, construction and implementation of measures of risk.

We will consider the **acceptability** regions of financial positions (and not their **optimality**).

Risk measure : rather a regulator's tool, or supervisor's.

Risk measure : defined as the requested cost to join the acceptability region.

Definition of exposures

- An *exposure* is described by a random variable X representing the discounted net *loss* of the exposure at the maturity time.
- Our aim is to quantify the risk of X by some number $\rho(X)$, where X belong to a given class $\mathcal{X}$ of financial or insurance positions.

Monetary risk measures

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a *monetary risk measure* if it satisfies the following conditions for all $X, Y \in \mathcal{X}$:

- $[MO]$, *Monotonicity* : if $X \geq Y$ then $\rho(X) \geq \rho(Y)$.
- $[TI]$, *Translation Invariance* or *Cash additivity* :
  $\rho(X - m) = \rho(X) - m, \; m \in \mathbb{R}$.
  This implies in particular :
  $\rho(X - \rho(X)) = \rho(X) - \rho(X) = 0$.

Monetary risk measures

- The financial meaning of monotonicity is clear: The downside risk of a position is increased if the loss profile is increased.
- Translation invariance is also called cash invariance. It is motivated by the interpretation of $\rho(X)$ as a capital requirement, i.e., $\rho(X)$ is the amount which should be added to the position X in order to make it acceptable. Thus, if the amount m is added to the position and invested in a risk-free manner, the capital requirement is reduced by the same amount.

Convex risk measures

A monetary risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is called a *convex risk measure* if it satisfies:

- [*CO*], *Convexity* : $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.

The axiom of convexity gives a precise meaning to the idea that diversification should not increase the risk.

Convex risk measures

If ρ is convex and normalized (i.e $\rho(0) = 0$) then:

- $\rho(\lambda X) \leq \lambda\rho(X)$, for $0 \leq \lambda \leq 1$.
- $\rho(\lambda X) \geq \lambda\rho(X)$, for $\lambda \geq 1$.

Thus, the axiom of convexity also gives a precise meaning to the idea that an increased exposure should increase the liquidity risk.

Risk appetite has a non linear effect on risk measurement!

Dominant practical approach

The Value-at-Risk is the main practical risk measure. Computing an α-quantile

- $\alpha =$ reference probability (acceptable bankruptcy probability).
- Losses values that are attained only with that probability.

$(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed underlying probability space.

$$VaR_\alpha(X) := q_X(\alpha) = \inf\{x \in \mathbb{R} \text{ such that } F_X(x) \geq \alpha\}$$

where F_X denotes the cumulative distribution function of a r.v X.

Average VaR

The *Average Value-at-Risk* at level $\alpha \in (0,1]$ of a position $X \in \mathcal{X}$ is given by:

$$AVaR_\alpha(X) = \frac{1}{\alpha}\int_{1-\alpha}^{1} q_X(u)du$$

$AVaR_\alpha$ is a coherent risk measure and we have:

$$AVaR_\alpha(X) = \bar{q} + \frac{1}{\alpha}E\left[(X-\bar{q})^+\right]$$

where $\bar{q} = q_X(1-\alpha)$.

Worst case risk measure

Consider the *worst case risk measure* ρ_{max} defined by:

$$\rho_{max}(X) = \sup_{\omega \in \Omega} X(\omega)$$

The value $\rho_{max}(X)$ is the least upper bound for the potential loss which can occur in any scenario. The corresponding acceptance set $\mathcal{A}$ is given by the convex cone of all non-negative functions in X. Thus, $\rho_{max}(X)$ is a coherent measure of risk. It is the most conservative measure of risk in the sense that any normalized monetary risk measure ρ on $\mathcal{X}$ satisfies:

$$\rho(X) \leq \rho(\sup_{\omega \in \Omega} X(\omega)) = \rho_{max}(X)$$

Entropic risk measure

Consider the *entropic risk measure* ρ_γ defined by:

$$\rho_\gamma(X) = \gamma \ln \mathbb{E}_{\mathbb{P}}[\exp(\frac{1}{\gamma}X)], \ \gamma \in \mathbb{R}^+$$

Interpretations:

- The value $\rho_\gamma(X)$ corresponds to a distorted mean of X, i.e to an integral value in the framework of Pap's g-calculus.
- The value $\rho_\gamma(X)$ is the certainty equivalent of the random exposure X, for an exponential utility function.
- We will see that $\rho_\gamma(X)$ is the worst mean value minus a penalty, evaluated through a family of models, the penalty being given by the entropy of the models.

Robust representation of convex risk measures

When ρ is a coherent risk measure (particular case of a convex risk measure), the representation takes the form:

$$\rho(X) = \max_{Q \in \mathcal{Q}} E_Q(X), \; X \in \mathcal{X}$$

What about pricing?

- Consider an (re-) insurance contract offering a protection $C(X)$ for an initial exposure X.
- The **indifference price** π of a possible buyer, using a risk measure ρ, is defined by

$$\rho(X - C + \pi) = \rho(X)$$

- In the most simple case we obtain

$$\pi = \rho(X) - \rho(X - C).$$

Distortion functions

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- Concave distortion = risk averse agent
- Convex distortion = risk seeking agent
- S-shaped distortion ?

Distortion functions

- The PH-transform (Proportional hasard) corresponds to $g(y) = y^r$, $r > 0$.
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C. Robert and P. Thrond, *Distortion risk measures, ambiguity aversion and optimal effort*, ASTIN Bulletin, 2014.

A possible measure of model uncertainty

Solve the following optimization problem :

$$\sup_{X \in \mathcal{L}_{\mu,\sigma}} \rho(X)$$

where $\mathcal{L}_{\mu,\sigma}$ denotes the set of probability laws on $\mathbb{R}$ with mean μ and variance σ^2, and where ρ is a law invariant risk measure.

Motivations

- Quantification of model uncertainty: Barrieu and Scandolo, *Assessing financial model risk* (2014)

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Proposed metric:

$$RM(X_0, \mathcal{L}) := \frac{\overline{\rho}(\mathcal{L}) - \rho(X_0)}{\overline{\rho}(\mathcal{L}) - \underline{\rho}(\mathcal{L})}$$

where

$$\overline{\rho}(\mathcal{L}) := \sup_{X \in \mathcal{L}} \rho(X) \quad \text{and} \quad \underline{\rho}(\mathcal{L}) := \inf_{X \in \mathcal{L}} \rho(X)$$

Motivations

- Model free pricing in insurance.

Compute

$$\sup_{X \in \mathcal{L}} \mathbb{E}[v(X)]$$

where v is a given convex function.

- Jansen, Haezendonck and Goovaerts (1986)
- Hurlimann (1988)

Methodology

We reformulate the problem in the following manner :

$$\sup_{q \in \mathcal{Q}_{\mu,\sigma}} \Phi(q)$$

where $\mathcal{Q}_{\mu,\sigma}$ denotes the set of quantile functions of probability laws on $\mathbb{R}$ with mean μ and variance σ^2, and where Φ is such that $\rho(X) = \Phi(q_X)$.

A result

Proposition

Assume that Φ is convex, then

$$\sup_{q \in \mathcal{Q}_{\mu,\sigma}} \Phi(q) = \sup_{q \in \mathcal{Q}^2_{\mu,\sigma}} \Phi(q)$$

where $Q^2_{\mu,\sigma}$ denotes the set of quantile functions of diatomic probability laws on $\mathbb{R}$ with mean μ and variance σ^2:

$$q(x) = K\mathbf{1}_{[0,c)}(x) + (K+\gamma)\mathbf{1}_{[c,1]}(x)$$

where $K \in \mathbb{R}$, $\gamma > 0$ and $c \in (0,1)$.

Application to DRM

Application to the case of distortion risk measures:

A distortion risk measure is **law invariant** and can be written

$$\Phi(\overline{q}) = \int_0^1 \overline{q}(u) d\psi(u)$$

where ψ is a given distortion function. It is a **linear** functional in the $\overline{q}$ variable !

Application to DRM

To obtain a superior bound, all one need to compute is:

$$\sup_{\overline{q}\in\overline{\mathcal{Q}}^2_{\mu,\sigma}} \Phi(\overline{q}) = \sup_{(K,\gamma,c)} \phi(K,\gamma,c)$$

$$= \sup_{c\in(0,1)} \psi(1-c)\left(\mu+\sigma\sqrt{\frac{c}{1-c}}\right) + (1-\psi(1-c))\left(\mu-\sigma\sqrt{\frac{1-c}{c}}\right)$$

Application to DRM

To obtain a superior bound, all one need to compute is:

$$\sup_{\overline{q}\in\overline{\mathcal{Q}}^2_{\mu,\sigma}} \Phi(\overline{q}) = \mu + \sigma\left(\sup_{c\in(0,1)} \frac{\psi(c)-c}{\sqrt{c(1-c)}}\right).$$

We can get rid of μ and σ.

Application to DRM

Corollary

$$\sup_{\overline{q} \in \overline{\mathcal{Q}}_{\mu,\sigma}} \int_0^1 \overline{q}(u) d\psi(u) < +\infty$$

if and only if

$$\lim_{x \to 0^+} \frac{\psi(x) - x}{\sqrt{x(1-x)}} < +\infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{\psi(x) - x}{\sqrt{x(1-x)}} < +\infty$$

Application to DRM

We can retrieve the following classical result:

For $\psi(u) := \mathbf{1}_{u \geq \alpha}$, $\alpha \in (0,1)$, we have

$$\sup_{X \in \mathcal{L}_{\mu,\sigma}} VaR_\alpha(X) = \mu + \sigma \sqrt{\frac{1-\alpha}{\alpha}}$$

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Free bonus:

$$\inf_{X \in \mathcal{L}_{\mu,\sigma}} VaR_\alpha(X) = \mu - \sigma \sqrt{\frac{\alpha}{1-\alpha}}$$